Stochastic Games with Partial Observation: Decidable and Undecidable Problems

Hugo Gimbert    Youssouf Oualhadj

Games 09
Udine, Italy

September 14, 2009
Probabilistic Automata: Decidable and Undecidable Problems

Hugo Gimbert    Youssouf Oualhadj

Games 09
Udine, Italy

September 14, 2009
Outline

Probabilistic Automata

The Emptiness Problem

The Threshold Isolation Problem

Conclusion
Definition

A probabilistic automaton \( A \) is defined by a tuple:

\[
A = (Q, \Sigma, M, q_0, F)
\]
Definition

A probabilistic automaton $\mathcal{A}$ is defined by a tuple: $\mathcal{A} = (Q, \Sigma, M, q_0, F)$
Computation on an input

Given a word $w \in \Sigma^*$, how we compute $\mathbb{P}_A(w)$?
Computation on an input

Given a word $w \in \Sigma^*$, how we compute $\mathbb{P}_A(w)$?
Computation on an input

Given a word $w \in \Sigma^*$, how we compute $P_A(w)$?

$w = ab$

$\delta_0 = (1, 0, 0, 0, 0, 0)$. 
Computation on an input

Given a word $w \in \Sigma^*$, how we compute $\mathbb{P}_A(w)$?

Let $w = ab$. Then

$\delta_1 = \delta_0.M_a$

and

$a, b = \left(0, \frac{3}{4}, \frac{1}{4}, 0, 0, 0\right)$.
Given a word $w \in \Sigma^*$, how we compute $\mathbb{P}_A(w)$?

$$w = ab$$

$\delta_2 = \delta_1 \cdot M_b$

$$a, b = \left(0, \frac{3}{8}, 0, \frac{3}{16}, \frac{5}{16}, \frac{1}{8}\right)$$
Given a word \( w \in \Sigma^* \), how we compute \( \mathbb{P}_A(w) \)?

\[
\mathbb{P}_A(w) = \frac{3}{16} + \frac{1}{8} = \frac{5}{16}
\]
Outline

Probabilistic Automata

The Emptiness Problem

The Threshold Isolation Problem

Conclusion
The emptiness problem

Accepted Language [Rabin, 63]
Let $A$ a probabilistic automaton:

$$L_A = \left\{ w \in \Sigma^* | \mathbb{P}_A \geq \frac{1}{2} \right\} .$$
The emptiness problem

Accepted Language [Rabin, 63]
Let $\mathcal{A}$ a probabilistic automaton:

$$
\mathcal{L}_\mathcal{A} = \left\{ w \in \Sigma^* | \mathbb{P}_\mathcal{A} \geq \frac{1}{2} \right\}.
$$

The emptiness problem

Given a probabilistic automata $\mathcal{A}$, decide if: $\mathcal{L}_\mathcal{A} \neq \emptyset$
The emptiness problem

Accepted Language [Rabin, 63]

Let $A$ a probabilistic automaton:

$$L_A = \left\{ w \in \Sigma^* | P_A \geq \frac{1}{2} \right\}.$$ 

The emptiness problem

Given a probabilistic automata $A$, decide if: $L_A \neq \emptyset$

Theorem [Paz, 71][Condon Lipton, 05]

The emptiness problem for probabilistic automata is undecidable.
The emptiness problem

Accepted Language [Rabin, 63]
Let $\mathcal{A}$ a probabilistic automaton:

$$\mathcal{L}_A = \left\{ w \in \Sigma^* \left| \mathbb{P}_A \geq \frac{1}{2} \right. \right\}.$$

The emptiness problem
Given a probabilistic automata $\mathcal{A}$, decide if: $\mathcal{L}_A \neq \emptyset$

Theorem [Paz, 71][Condon Lipton, 05]
The emptiness problem for probabilistic automata is undecidable.

Corollary
The following problem is undecidable:
Given a non deterministic automaton is there a word such that at least half the computations are accepting?
New undecidability proof

Lemma

Given a probabilistic automaton $\mathcal{A}$, it is undecidable whether there exists a word $w \in \Sigma^*$ such that $\mathbb{P}_{\mathcal{A}}(w) = \frac{1}{2}$
New undecidability proof

Lemma

Given a probabilistic automaton $A$, it is undecidable whether there exists a word $w \in \Sigma^*$ such that $P_A(w) = \frac{1}{2}$

Proof.

Let $(u_i, v_i)_{i \in I}$ and $u_i, v_i \in \{0, 1\}^*$.
New undecidability proof

Lemma

Given a probabilistic automaton $\mathcal{A}$, it is undecidable whether there exists a word $w \in \Sigma^*$ such that $P_{\mathcal{A}}(w) = \frac{1}{2}$

Proof.

Let $(u_i, v_i)_{i \in I}$ and $u_i, v_i \in \{0, 1\}^*$.

$\forall w_i \in \Sigma : f(w_1 \ldots w_n) = \frac{w_n}{2} + \cdots + \frac{w_1}{2^n}$. 
New undecidability proof

Lemma

*Given a probabilistic automaton \( \mathcal{A} \), it is undecidable whether there exists a word \( w \in \Sigma^* \) such that \( P_{\mathcal{A}}(w) = \frac{1}{2} \)*

Proof.

Let \((u_i, v_i)_{i \in I}\) and \( u_i, v_i \in \{0, 1\}^* \).

\[ \forall w_i \in \Sigma : f(w_1 \ldots w_n) = \frac{w_n}{2} + \cdots + \frac{w_1}{2^n} \]

We construct \( \mathcal{A}_u \) and \( \mathcal{A}_v \) such that:
New undecidability proof

Lemma

*Given a probabilistic automaton \( A \), it is undecidable whether there exists a word \( w \in \Sigma^* \) such that \( \mathbb{P}_A(w) = \frac{1}{2} \)*

Proof.

Let \( (u_i, v_i)_{i \in I} \) and \( u_i, v_i \in \{0, 1\}^* \).

\[ \forall w_i \in \Sigma : f(w_1 \ldots w_n) = \frac{w_n}{2} + \cdots + \frac{w_1}{2^n} . \]

We construct \( A_u \) and \( A_v \) such that:

\[ \mathbb{P}_{A_u}(i_1 \ldots i_k) = f(u_{i_1} \ldots u_{i_k}) . \]
New undecidability proof

Lemma

Given a probabilistic automaton $\mathcal{A}$, it is undecidable whether there exists a word $w \in \Sigma^*$ such that $\mathbb{P}_{\mathcal{A}}(w) = \frac{1}{2}$

Proof.

Let $(u_i, v_i)_{i \in I}$ and $u_i, v_i \in \{0, 1\}^*$.

$\forall w_i \in \Sigma : f(w_1 \ldots w_n) = \frac{w_n}{2} + \cdots + \frac{w_1}{2^n}$.

We construct $\mathcal{A}_u$ and $\mathcal{A}_v$ such that:

- $\mathbb{P}_{\mathcal{A}_u}(i_1 \ldots i_k) = f(u_{i_1} \ldots u_{i_k})$.
- $\mathbb{P}_{\mathcal{A}_v}(i_1 \ldots i_k) = f(v_{i_1} \ldots v_{i_k})$. 
New undecidability proof

Lemma

Given a probabilistic automaton $A$, it is undecidable whether there exists a word $w \in \Sigma^*$ such that $P_A(w) = \frac{1}{2}$

Proof.

Let $(u_i, v_i)_{i \in I}$ and $u_i, v_i \in \{0, 1\}^*$.

$\forall w_i \in \Sigma : f(w_1 \ldots w_n) = \frac{w_n}{2} + \cdots + \frac{w_1}{2^n}$.

We construct $A_u$ and $A_v$ such that:

$\triangleright P_{A_u}(i_1 \ldots i_k) = f(u_{i_1} \ldots u_{i_k})$.

$\triangleright P_{A_v}(i_1 \ldots i_k) = f(v_{i_1} \ldots v_{i_k})$.

Let $A : \forall w \in \Sigma^* : P_A(w) = \frac{1}{2} P_{A_u}(w) + \frac{1}{2} (1 - P_{A_v}(w))$. 
New undecidability proof

Lemma
Given a probabilistic automaton $A$, it is undecidable whether there exists a word $w \in \Sigma^*$ such that $\mathbb{P}_A(w) = \frac{1}{2}$

Proof.
Let $(u_i, v_i)_{i \in I}$ and $u_i, v_i \in \{0, 1\}^*$.
\[ \forall w_i \in \Sigma : f(w_1 \ldots w_n) = \frac{w_n}{2} + \cdots + \frac{w_1}{2^n} . \]
We construct $A_u$ and $A_v$ such that:

- $\mathbb{P}_{A_u}(i_1 \ldots i_k) = f(u_i_1 \ldots u_i_k)$ .
- $\mathbb{P}_{A_v}(i_1 \ldots i_k) = f(v_i_1 \ldots v_i_k)$ .

Let $A : \forall w \in \Sigma^* : \mathbb{P}_A(w) = \frac{1}{2} \mathbb{P}_{A_u}(w) + \frac{1}{2} (1 - \mathbb{P}_{A_v}(w)) .

PCP \iff \exists i_1 \ldots i_k : \mathbb{P}_A(i_1 \ldots i_k) = \frac{1}{2} . \quad \square
New undecidability proof

Lemma

Given a probabilistic automaton \(\mathcal{A}\), it is undecidable whether there exists a word \(w \in \Sigma^*\) such that \(P_{\mathcal{A}}(w) = \frac{1}{2}\)

Proof.

Let \((u_i, v_i)_{i \in I}\) and \(u_i, v_i \in \{0, 1\}^*\). \(\forall w_i \in \Sigma: f(w_1 \ldots w_n) = \frac{w_n}{2} + \cdots + \frac{w_1}{2^n}\).

We construct \(\mathcal{A}_u\) and \(\mathcal{A}_v\) such that:

\[\begin{align*}
\mathcal{A}_u &: \forall w \in \Sigma^*: P_{\mathcal{A}_u}(w) = f(u_1 \ldots u_k), \\
\mathcal{A}_v &: \forall w \in \Sigma^*: P_{\mathcal{A}_v}(w) = f(v_1 \ldots v_k).
\end{align*}\]

Let \(\mathcal{A}: \forall w \in \Sigma^*: P_{\mathcal{A}}(w) = \frac{1}{2} P_{\mathcal{A}_u}(w) + \frac{1}{2} (1 - P_{\mathcal{A}_v}(w))\).

\[\text{PCP} \iff \exists i_1 \ldots i_k: P_{\mathcal{A}}(i_1 \ldots i_k) = \frac{1}{2}.\]

Lemma

Let \(\mathcal{A}\) a probabilistic automaton, there exists \(\mathcal{A}'\) such that:

\[\left(\exists w \in \Sigma^*: P_{\mathcal{A}}(w) = \frac{1}{2}\right) \iff \left(\exists w \in \Sigma^*: P_{\mathcal{A}'}(w) \geq \frac{1}{2}\right)\]
Some decidability results

Proposition [Gimbert, O.]

The emptiness problem for automata with 1 probabilistic state is decidable.
Some decidability results

Proposition [Gimbert, O.]
The emptiness problem for automata with 1 probabilistic state is decidable.

Proposition [Gimbert, O.]
The emptiness problem for automata with 2 probabilistic states is undecidable.
Some decidability results

Proposition [Gimbert, O.]
The emptiness problem for automata with 1 probabilistic state is decidable.

Proposition [Gimbert, O.]
The emptiness problem for automata with 2 probabilistic states is undecidable.

Proof.
We transform any probabilistic automaton $\mathcal{A}$ with input in $\Sigma$ to a new automaton $\mathcal{A}'$ with input in $\Sigma'$ such that:

\[ P_{\mathcal{A}}(w) = \lambda \iff \exists w' \in \Sigma' : P_{\mathcal{A}'}(w') = \frac{1}{2} \lambda + \frac{1}{2}. \]
Some decidability results

Proposition [Gimbert, O.]
The emptiness problem for automata with 1 probabilistic state is decidable.

Proposition [Gimbert, O.]
The emptiness problem for automata with 2 probabilistic states is undecidable.

Proof.
We transform any probabilistic automaton $A$ with input in $\Sigma$ to a new automaton $A'$ with input in $\Sigma'$ such that:

- $A'$ has 2 probabilistic states.
Some decidability results

Proposition [Gimbert, O.]
The emptiness problem for automata with 1 probabilistic state is decidable.

Proposition [Gimbert, O.]
The emptiness problem for automata with 2 probabilistic states is undecidable.

Proof.
We transform any probabilistic automaton $A$ with input in $\Sigma$ to a new automaton $A'$ with input in $\Sigma'$ such that:

- $A'$ has 2 probabilistic states.
- $\forall w \in \Sigma^*, \forall \lambda > 0$:

\[
\mathbb{P}_A(w) = \lambda \iff \exists w' \in \Sigma'^*: \mathbb{P}_{A'}(w') = \frac{1}{2} \lambda + \frac{1}{2}.
\]
Outline

Probabilistic Automata

The Emptiness Problem

The Threshold Isolation Problem

Conclusion
The Threshold Isolation Problem

Definition

Let $\mathcal{A}$ a probabilistic automaton. $\lambda$ is isolated with respect to $\mathcal{A}$ if:

$$\exists \varepsilon > 0, \forall w \in \Sigma^*: |P_{\mathcal{A}}(w) - \lambda| \geq \varepsilon$$
The Threshold Isolation Problem

Definition
Let $A$ a probabilistic automaton. $\lambda$ is isolated with respect to $A$ if:

$$\exists \epsilon > 0, \forall w \in \Sigma^*: |P_A(w) - \lambda| \geq \epsilon$$

Theorem [Bertoni, 77]
Given a probabilistic automaton $A$ and a rational $0 < \lambda < 1$, it is undecidable whether if $\lambda$ isolated with respect to $A$ or not.
The Threshold Isolation Problem

Definition
Let $\mathcal{A}$ a probabilistic automaton. $\lambda$ is isolated with respect to $\mathcal{A}$ if :

$$\exists \varepsilon > 0, \forall w \in \Sigma^* : |P_{\mathcal{A}}(w) - \lambda| \geq \varepsilon$$

Theorem [Bertoni, 77]
Given a probabilistic automaton $\mathcal{A}$ and a rational $0 < \lambda < 1$, it is undecidable whether if $\lambda$ isolated with respect to $\mathcal{A}$ or not.

Open problem
What if $\lambda = 1$?
The Threshold Isolation Problem

Definition
Let $\mathcal{A}$ a probabilistic automaton. $\lambda$ is isolated with respect to $\mathcal{A}$ if:
\[
\exists \varepsilon > 0, \forall w \in \Sigma^* : |\mathbb{P}_\mathcal{A}(w) - \lambda| \geq \varepsilon
\]

Theorem [Bertoni, 77]
Given a probabilistic automaton $\mathcal{A}$ and a rational $0 < \lambda < 1$, it is undecidable whether if $\lambda$ isolated with respect to $\mathcal{A}$ or not.

Open problem
What if $\lambda = 1$?

Theorem [Gimbert, O.]
Given a probabilistic automaton $\mathcal{A}$, it is undecidable whether if $1$ is isolated with respect to $\mathcal{A}$ or not.
Sketch of proof

Reduction of the emptiness problem

![Diagram]

1.\(a, x\) to 1
2.\(a, 1 - x\) to 2
3.\(b\) to 3
4.\(a\) to 4
5.\(a, 1 - x\) to 5
6.\(a, x\) to 6
7.\(b\) to 6
8.\(a, b\) to 7

The state machine transitions are as follows:
- From state 1, on input \(a\), transition to state 2.
- From state 2, on input \(1 - x\), transition back to state 1.
- From state 1, on input \(b\), transition to state 3.
- From state 3, on input \(a\), transition to state 4.
- From state 4, on input \(1 - x\), transition back to state 5.
- From state 5, on input \(a\), transition to state 6.
- From state 6, on input \(b\), transition back to state 7.
- From state 7, on input \(a, b\), transition to state 8.
Sketch of proof

Reduction of the emptiness problem

Lemma

1 is isolated with respect to A iff $x > \frac{1}{2}$. 
Sketch of proof

Reduction of the emptiness problem

![Diagram](image-url)
Sketch of proof

Reduction of the emptiness problem

Lemma

1 is isolated with respect to $A$ iff $L_B \neq \emptyset$. 
Outline

Probabilistic Automata

The Emptiness Problem

The Threshold Isolation Problem

Conclusion
Conclusion

New Results

- New proof for the emptiness problem.
Conclusion

New Results

- New proof for the emptiness problem.
- Decidable for 1 probabilistic state.
Conclusion

New Results

- New proof for the emptiness problem.
- Decidable for 1 probabilistic state.
- Undecidable for 2 probabilistic states.
Conclusion

New Results

- New proof for the emptiness problem.
- Decidable for 1 probabilistic state.
- Undecidable for 2 probabilistic states.
- Undecidability of the threshold problem when $\lambda = 1$ and $\lambda = 0$. 
Conclusion

New Results

- New proof for the emptiness problem.
- Decidable for 1 probabilistic state.
- Undecidable for 2 probabilistic states.
- Undecidability of the threshold problem when $\lambda = 1$ and $\lambda = 0$.

Agenda

- Search new class of automata for which the emptiness problem is decidable.
Conclusion

New Results

- New proof for the emptiness problem.
- Decidable for 1 probabilistic state.
- Undecidable for 2 probabilistic states.
- Undecidability of the threshold problem when $\lambda = 1$ and $\lambda = 0$.

Agenda

- Search new class of automata for which the emptiness problem is decidable.
- Find class of games with partial observations whose values are computable.