A Class of Probabilistic Automata Whose Value-One Problem is Decidable

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\{3\} is the set of accepting states

Formal definition

A probabilistic automaton is a tuple $A = (Q, \mathcal{A}, (M_a)_{a \in \mathcal{A}}, q_0, F)$. 
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$P_{A(aaab)} = 1$ is the acceptance probability.
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Probabilistic Automata

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A bit of history

Given a probabilistic automaton $\mathcal{A}$, and a rational $0 < \lambda < 1$
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Emptiness problem (Rabin 63)
Is there a word \( w \in \mathcal{A}^* \) such that

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P_{\mathcal{A}}(w) \geq \lambda.
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Is there a threshold $\varepsilon > 0$ such that

$$\forall w \in A^* \ |\mathbb{P}_A(w) - \lambda| \geq \varepsilon .$$

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Is the special case of the isolation problem when $\lambda = 1$. 
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Is the special case of the isolation problem when $\lambda = 1$. 
Value-one problem

Definition
Let $\mathcal{A}$ a probabilistic automaton. $\mathcal{A}$ has the value 1 if:

$$\forall \varepsilon > 0, \exists w \in A^*, \mathbb{P}_\mathcal{A}(w) \geq 1 - \varepsilon.$$
Value-one problem

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Let $A$ a probabilistic automaton. $A$ has the value 1 if:

$$\forall \varepsilon > 0, \exists w \in A^*, P_A(w) \geq 1 - \varepsilon.$$ 

Theorem (Gimbert, O. 2009)

*The value 1 problem is undecidable.*

Corollary
Given a non-deterministic automaton on finite words, it is undecidable whether there exist words such that the proportion of rejected runs over all possible computations is arbitrarily small.

Theorem (Gimbert, O. 2010)

The value-one problem is decidable for $\#$-acyclic automata.
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Theorem (Gimbert, O. 2010)
The value-one problem is decidable for #-acyclic automata.
Definition (\$\#\$-acyclic)

\$A\$ is \$\#\$-acyclic iff the associated support graph \$G_A\$ has no cycle except self-loops.
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Definition (Support graph)
$G_A$ constructed from $A$ the following way:

\[
\begin{align*}
G_A & \quad 1 \quad a, \frac{1}{2} \quad 2 \\
    & \quad a, \frac{1}{2} \\
\end{align*}
\]
#-acyclic automata

**Definition ( #-acyclic)**

\( \mathcal{A} \) is #-acyclic iff the associated support graph \( G_{\mathcal{A}} \) has no cycle except self-loops.

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#-acyclic automata

**Definition ( #-acyclic)**

$\mathcal{A}$ is #-acyclic iff the associated support graph $\mathcal{G}_\mathcal{A}$ has no cycle except self-loops.

**Definition (Support graph)**

$\mathcal{G}_\mathcal{A}$ constructed from $\mathcal{A}$ the following way:

\[ 1 \xrightarrow{a, \frac{1}{2}} 2 \xrightarrow{a, \frac{1}{2}} 1 \]

\[ 2 \xrightarrow{a} 1 \]

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Examples
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\begin{itemize}
\item \{1, 2\}
\item \{1\} \quad \{2\}
\item \{1, 3\} \quad \{1, 2, 3, 4\} \quad \{2, 4\}
\item \{3\} \quad \{4\}
\item \{3, 4\}
\end{itemize}
Examples
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\[
\begin{align*}
\{1, 3\} & \quad \xrightarrow{b} \quad \{1, 2, 3, 4\} \\
\{1\} & \quad \xrightarrow{a} \quad \{2\}
\end{align*}
\]

\[
\begin{align*}
\{3\} & \quad \xrightarrow{a^\#} \quad \{3, 4\}
\end{align*}
\]
Examples

\[
\begin{align*}
\{1, 2\} & \quad a \quad \{1\} \\
\{1, 2, 3, 4\} & \quad b \quad \{2\} \\
\{3, 4\} & \quad a \quad \{3\} \\
\{3, 4\} & \quad b \quad \{4\}
\end{align*}
\]
Why is it decidable?

Definition

- Reachability in the support graph is called $\#\text{-reachability}$.
- A set $T$ is said to be limit-reachable from another set $S$ if there exists a sequence $w_0, w_1, \cdots \in A^*$ such that:

$$\forall s \in S, \ P_A(s \xrightarrow{w_n} T) \xrightarrow{n \to \infty} 1.$$
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Key-property

In $\#$-acyclic automata:

$$\text{limit-reachability} \iff \#\text{-reachability}.$$
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Key-property

In \( \# \)-acyclic automata:

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\text{limit-reachability } \iff \# \text{-reachability}.
\]

\[
(val_A = 1) \iff F \text{ is } \# \text{-reachable from } I.
\]
From ♯-path to limit path
From $\#$-path to limit path

\[
\begin{align*}
\{2\} & \xrightarrow{a} \{1, 2, 3\} & \{1, 2\} & \xleftarrow{a} \{1\} \\
\{3\} & \xrightarrow{b} \{2, 3\} \xrightarrow{a} \{1, 2, 3\} & \{2, 3\} & \xrightarrow{b} \{1, 2\} \\
\{1, 3\} & \xrightarrow{b} \{1, 2\} & \{1, 3\} & \xrightarrow{a} \{1, 2, 3\}
\end{align*}
\]
From $\#$-path to limit path
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$$\mathbb{P}_A(ab^n) = 1 - \frac{1}{2^{n+2}}.$$
From limit path to $\#$-path

Flooding lemma

Assume

$$\forall a \in A, \ Q \cdot a = Q \cdot a^\# = Q.$$

Then, $Q$ is the unique set
limit-reachable support from $Q$. 

![Diagram](image-url)
From limit path to $\#$-path

Flooding lemma
Assume
\[ \forall a \in A, \ Q \cdot a = Q \cdot a^\# = Q. \]

Then, $Q$ is the unique set limit-reachable support from $Q$.

Leaf lemma
There exists a unique leaf $S$ $\#$-reachable from $Q$. Every set limit-reachable from $Q$ contains $S$. 
Inductive step

If $T$ is limit-reachable from $S_0$ then either:

- $S_0 = T$.
- $\exists S_1 \neq S_0$ s.t $S_1$ is $\#_\ell$-reachable from $S_0$ and $T$ is limit-reachable from $S_1$. 
Inductive step
If $T$ is limit-reachable from $S_0$ then either:

► $S_0 = T$.

► $\exists S_1 \neq S_0$ such that $S_1$ is $\#^*$-reachable from $S_0$ and $T$ is limit-reachable from $S_1$.

Proof.
Let $(u_n)_{n \in \mathbb{N}}$ be a limit-path from $S_0$ to $T$. 

\[
\begin{array}{c}
\hspace{1cm} u_0 \hspace{1cm} \\
\hspace{1cm} \hspace{4cm} \hspace{1cm} \\
\hspace{1cm} u_1 \hspace{1cm} \\
\hspace{1cm} \hspace{4cm} \hspace{1cm} \\
\hspace{1cm} \vdots \\
\hspace{1cm} \hspace{4cm} \hspace{1cm} \\
\hspace{1cm} \vdots \\
\hspace{1cm} \hspace{4cm} \hspace{1cm} \\
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\end{array}
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Inductive step

If \( T \) is limit-reachable from \( S_0 \) then either:

1. \( S_0 = T \).
2. \( \exists S_1 \neq S_0 \) s.t \( S_1 \) is \( \# \)-reachable from \( S_0 \) and \( T \) is limit-reachable from \( S_1 \).

Proof.

Let \( (u_n)_{n \in \mathbb{N}} \) be a limit-path from \( S_0 \) to \( T \).

Let \( A_0 = \{ a \in A \mid S_0 \cdot a = S_0 \} \).
Inductive step

If $T$ is limit-reachable from $S_0$ then either:

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Let $A_0 = \{a \in A \mid S_0 \cdot a = S_0\}$. 

\[ S_0 \xrightarrow{(\delta S_0 \cdot v_n)} \delta \xrightarrow{w_n} T \]
Inductive step

If \( T \) is limit-reachable from \( S_0 \) then either:

- \( S_0 = T \).
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Let \( A_0 = \{ a \in A \mid S_0 \cdot a = S_0 \} \).
\[ \text{Supp}(\delta) = S_0: \ S_1 = \text{Supp}(\delta) \cdot b. \]
**Inductive step**

If $T$ is limit-reachable from $S_0$ then either:

- $S_0 = T$.
- $\exists S_1 \neq S_0$ s.t $S_1$ is ♯-reachable from $S_0$ and $T$ is limit-reachable from $S_1$.

**Proof.**

Let $(u_n)_{n \in \mathbb{N}}$ be a limit-path from $S_0$ to $T$.

Let $A_0 = \{ a \in A \mid S_0 \cdot a = S_0 \}$.

Supp$(\delta) = S_0$: $S_1 = \text{Supp}(\delta) \cdot b$.

Supp$(\delta) \neq S_0$: Apply the leaf lemma to $\mathcal{A}[S_0, A_0]$, let $S_1$ the unique ♯-reachable leaf.
Conclusion

- Find larger class of probabilistic automata for which the value-one problem is decidable.
- Extend this result to the general case of stochastic games of imperfect informations.