Prefix Rewriting and the Pushdown Hierarchy

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Reachability Problem

Overview

1. Prefix Rewriting and the reachability problem
2. Interpretations
3. Unfoldings and Muchnik's Theorem
4. The pushdown hierarchy

Prefix Rewriting and the Reachability Problem

Rewriting Over Words

Rewriting system: Finite set S of rules $u \to v$

Different uses of a rule $u \to v$ for the rewrite relation $\vdash$

- **Infix rewriting: $xuy \vdash xvy$**
- **Post's canonical systems: $ux \vdash xv$**
- **Prefix rewriting (Büchi's regular canonical systems): $ux \vdash vx$**

Fundamental results:

Infix rewriting systems and Post's canonical systems allow to simulate Turing machines.

Büchi 1965: Prefix rewriting systems generate regular sets from regular sets of "axioms", and the derivability relation is decidable.

The Setting of Pushdown Automata

A pushdown automaton has the form $\mathcal{P} = (P, \Sigma, \Gamma, p_0, Z_0, \Delta)$

Configurations are words from $P\Gamma^*$

A transition induces a move from $p\gamma w$ to quw

Write $p\gamma w \vdash quw$

So pushdown automata are a special from of prefix rewriting systems.

Consequence of Büchi's Theorem:

The reachable configurations of a pushdown automaton form a regular set.

The Reachability Sets

Given a pushdown automaton $\mathcal{P} = (P, \Sigma, \Gamma, p_0, Z_0, \Delta)$ and $T \subseteq P\Gamma^*$

$$\text{pre}^*(T) := \{pv \in P\Gamma^* \mid \exists qw \in T : pv \vdash^* qw\}$$

Analogously $\text{post}^*(T)$.

We may suppress Σ and q_0, Z_0 and obtain a "pusdown system $\mathcal{P} = (Q, \Gamma, \Delta)$ with transitions of the form (p, γ, v, q).

Given a pushdown system $\mathcal{P} = (P, \Gamma, \Delta)$ and a finite automaton recognizing a set $T \subseteq P\Gamma^*$, one can compute a finite automaton recognizing $\text{pre}^*(T)$, similarly for $\text{post}^*(T)$.

Deciding $p_1 w_1 \vdash^* p_2 w_2$:

Set $T = \{p_2 w_2\}$ and check whether the automaton recognizing $\text{pre}^*(T)$ accepts $p_1 w_1$.

Example

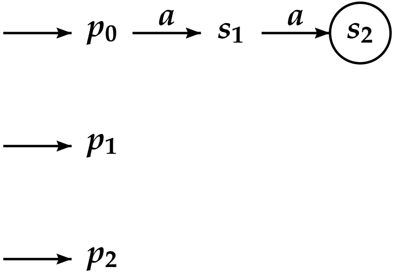
$\mathcal{P} = (P, \Gamma, \Delta)$ **with** $P = \{p_0, p_1, p_2\}$, $\Gamma = \{a, b, c\}$,

$\Delta =$
$\{(p_0 a \rightarrow p_1 ba), (p_1 b \rightarrow p_2 ca), (p_2 c \rightarrow p_0 b), (p_0 b \rightarrow p_0)\}$

$T = \{p_0 aa\}$.

P-automaton for T:

$\mathcal{A}$:



Saturation Algorithm: Idea

rule: $p\gamma \to qv$

$qvw \in pre^*(T)$

$p\gamma w$

So $p\gamma w \in pre^*(T)$

$\mathfrak{A}: q \xrightarrow{v} r \xrightarrow{w} F$

Add transition (p, γ, r) to $\mathfrak{A}$

Then $\mathfrak{A}: p \xrightarrow{a} r \xrightarrow{w} F$

Saturation Algorithm

Input: P-automaton $\mathcal{A}$, pushdown system $\mathcal{P} = (P, \Gamma, \Delta)$

$\mathcal{A}_0 := \mathcal{A}$, $i := 0$

REPEAT:

  IF $pa \to p'v \in \Delta$ and $\mathcal{A}_i : p' \xrightarrow{v} q$ **THEN**

    add (p, a, q) to $\mathcal{A}_i$ and obtain $\mathcal{A}_{i+1}$

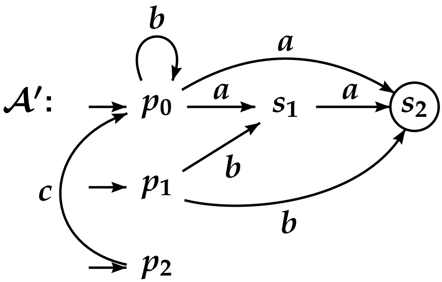
    $i := i + 1$

UNTIL no transition can be added

$\overline{\mathcal{A}} := \mathcal{A}_i$

Output: $\mathcal{A}'$

Example: Result



So for $T = \{p_0 aa\}$:

$$pre^*(T) = \; p_0 b^*(a + aa) \; + \; p_1 b \; + \; p_1 ba \; + \; p_2 cb^*(a + aa)$$

Alternative: Work in the Tree of Words

Consider a prefix rewriting system over $\{0,1\}$.

Convert prefix rewriting to suffix rewriting.

Then a rewrite step is definable in S2S.

Example: Rule $R : 11 \to 0$ leads from a word $w11$ to $w0$

Defining formula $\varphi_R(z,z')$: $\exists x(z = x11 \wedge z' = x0)$

For a system S let $\varphi_S(z,z') := \bigvee_{R \in S} \varphi_R(z,z')$

Preservation of Regularity

Let $L \subseteq \{0,1\}^*$ **be regular.**

There is an S2S-formula $\varphi_L(x)$ **defining** L **in the tree** T_2

We can write $L \subseteq Y$ **for** $\forall y(\varphi_L(y) \rightarrow Y(y))$

Then $x \in \mathrm{post}^*(L)$ **iff**

$\forall Y[(L \subset Y$ **and** $\forall z, z'(Y(z) \wedge \varphi_S(z,z')) \rightarrow Y(z')) \rightarrow Y(x)]$

The formula $\psi(X): \ \forall x(X(x) \leftrightarrow \text{“}x \in \mathrm{post}^*(L)\text{''})$ **is satisfied by a unique set.**

By Rabin's Basis Theorem it must be regular.

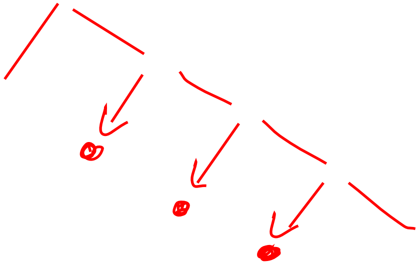
Interpretations

A First Example

Show Rabin's Tree Theorem for $T_3 = (\{0,1,2\}^*, S_0^3, S_1^3, S_2^3)$.

Idea: Obtain a copy of T_3 in T_2:

Consider T_2-vertices in $T = (10 + 110 + 1110)^*$.



Interpretation: Details

The element $i_1 \ldots i_m$ of T_3 is coded by

$1^{i_1+1}0 \ldots 1^{i_m+1}0$ in T_2.

Define the set of codes by

$\varphi(x)$: "x is in the closure of ε under 10-, 110-, and 1110-successors"

Define the 0-th, 1-st 2-nd successors by $\psi_0(x, y), \psi_1(x, y), \psi_2(x, y)$

The structure $(\varphi^{T_2}, (\psi_i^{T_2})_{i=0,1,2})$ restricted to φ^{T_2} is isomorphic to T_3.

Interpretations in General

An MSO-interpretation of a structure $\mathcal{A} = (A, R^{\mathcal{A}}, \ldots)$ in a structure $\mathcal{B}$ is given by

- **a "domain formula" $\varphi(x)$**
- **for each relation $R^{\mathcal{A}}$ of $\mathcal{A}$, say of arity m, an MSO-formula $\psi(x_1, \ldots, x_m)$**

such that $\mathcal{A}$ is isomorphic to $(\varphi^{\mathcal{B}}, \psi^{\mathcal{B}}, \ldots)$

Then there is a transformation OF MSO-sentenceS χ (in the signature of $\mathcal{A}$) to sentences χ' (in the signature of $\mathcal{B}$) such that
$\mathcal{A} \models \chi$ iff $\mathcal{B} \models \chi'$.

Consequence:

If $\mathcal{A}$ is MSO-interpretable in $\mathcal{B}$ and the MSO-theory of $\mathcal{B}$ is decidable, then so is the MSO-theory of $\mathcal{A}$.

Pushdown Graphs

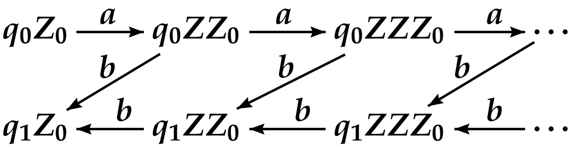
Consider $\mathcal{A}$ for language $L = \{a^n b^n \mid n \geq 0\}$:

$\mathcal{A} = (\{q_0, q_1\}, \{a, b\}, \{Z_0, Z\}, q_0, Z_0, \Delta)$ **with**

$$\Delta = \left\{ \begin{array}{ll} (q_0, Z_0, a, q_0, ZZ_0), & (q_0, Z, a, q_0, ZZ), \\ (q_0, Z, b, q_1, \varepsilon), & (q_1, Z, b, q_1, \varepsilon) \end{array} \right\}$$

Initial and final configuration: $q_0 Z_0$

The associated pushdown graph (of reachable configurations only) is:



Interpretation: Second Example

A pushdown graph is MSO-interpretable in T_2

Given pushdown automaton $\mathcal{A}$ with stack alphabet $\{1, \dots, k\}$ and states $q_1, \dots, q_m$.

Let $G_{\mathcal{A}} = (V_{\mathcal{A}}, E_{\mathcal{A}})$ be the corresponding PD graph.
$n := \max\{k, m\}$

Find an MSO-interpretation of $G_{\mathcal{A}}$ in T_n.

Represent configuration $(q_j, i_1 \dots i_r)$ by the vertex $i_r \dots i_1 j$.

$\mathcal{A}$-steps lead to local moves in T_n.

E.g. a push step from vertex $i_r \dots i_1 j$ to $i_r \dots i_1 i_0 j'$.

These edges are easily definable in MSO.

Hence: The MSO-theory of a PD graph is decidable.

Prefix-Recognizable Graphs

Instead of rules $u \to v$ we have rules $U \to \Upsilon$ wuth regular sets U, V.

Instead of describing a move from one word wu_0 to one wv_0

describe all admissible moves from a word wu to a word wv for a rule $U \to V$ with $u \in U, v \in V$.

This can be done by describing successful runs of the automata $\mathcal{A}_U, \mathcal{A}_V$ on the path segments from w to wu and from w to wv.

A graph is MSO-interpretable in T_2 iff its is prefix-recognizable.

Unfolding and Muchnik's Theorem

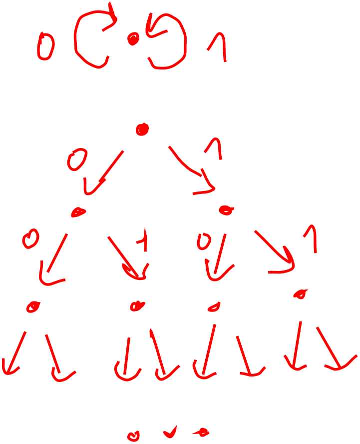
Unfoldings

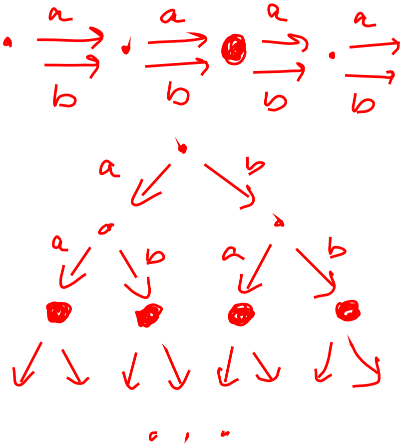
Given a graph $(V, (E_a)_{a \in \Sigma}, (P_b)_{b \in \Sigma'})$

the unfolding of G **from a given vertex** v_0 **is the following tree** $T_G(v_0) = (V', (E'_a)_{a \in \Sigma}, (P'_b)_{b \in \Sigma'})$**:**

- V' **consists of the vertices** $v_0 a_1 v_1 \ldots a_r v_r$ **with** $(v_{i-1}, v_i) \in E_{a_i}$**,**
- E'_a **contains the pairs** $(v_0 a_1 v_1 \ldots a_r v_r, v_0 a_1 v_1 \ldots a_r v_r a v)$ **with** $(v_r, v) \in E_a$**,**
- P'_b **the vertices** $v_0 a_1 v_1 \ldots a_r v_r$ **with** $v_r \in P_b$**.**

Examples





Unfolding Preserves Decidability

Theorem (Muchnik, Courcelle/Walukiewicz)

If the MSO-theory of G is decidable and v_0 is an MSO-definable vertex of G, then the MSO-theory of $T_G(v_0)$ is decidable.

We sketch the proof for pushdown graphs.

Their unfoldings are the “algebraic trees”.

Proof Architecture

Given an unfolding T of a pushdown graph G.

T is finitely branching, with labels say in Σ inherited from G.

For each MSO-formula $\varphi(X_1, \ldots, X_n)$ find a parity tree automaton $\mathcal{A}_\varphi$ such that

$\mathcal{A}_\varphi$ accepts $T(P_1, \ldots, P_n)$ iff $T[P_1, \ldots, P_n) \models \varphi(X_1, \ldots, X_n)$

The construction of the $\mathcal{A}\varphi$ follows precisely the pattern of Rabin's equivalence theorem.

Essential: In the complementation step we use the finite out-degree of G.

The general case is more involved.

Muchnik's Theorem: Continued

Result:

For a sentence φ we obtain a tree automaton $\mathcal{A}_\varphi$, say with state set Q and transition set Δ, with

$\mathcal{A}_\varphi$ accepts T iff $T \models \varphi$

The left-hand side says:

Automaton has a positional winning strategy in the associated game $\Gamma_{\mathcal{A},T}$

If $G = (V, E, v_0)$ for simplicity, the game graph consists of vertices

- **in $V \times Q$ (for Automaton)**
- **in $V \times \Delta$ (for Pathfinder)**

Muchnik's Theorem Finished

The game $\Gamma_{\mathcal{A},T}$ is played on a graph

$$G' = (V \times \{1, \ldots, k\}, E', (v_0, 1))$$

We use the following fact (shown next Friday):

The set of vertices v from where Player Automaton wins in the parity game over $G' = (V', E', v')$ is MSO-definable by a formula $\chi(x)$.

Translation Theorem:

For each sentence φ we can build a sentence φ^+ such that

$G' \models \varphi$ **iff** $G \models \varphi^+$

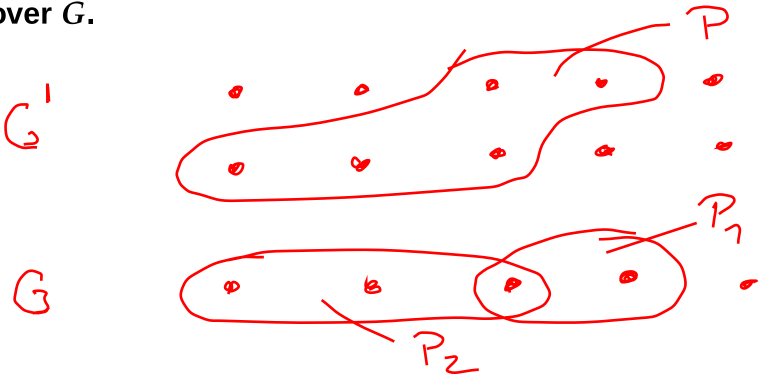
Since the MSO theory of G is decidable, we can decide the left-hand side.

Final Step

How to infer decidability of MTh$(G \times \{1,2\})$ from decidability of MTh(G)?

We do not address the definition of the edge relation but just give the idea:

Simulate a set quantifier over $G \times \{1,2\}$ by two set quantifiers over G.



Pushdown Hierarchy

Caucal's Proposal

We have now two processes which preserve decidability of MSO-theory:

- **interpretation (transforming a tree into a graph)**
- **unfolding (transforming a graph into a tree)**

Let us apply them in alternation!

We obtain the Caucal hierarchy or pushdown hierarchy.

Definition

- $\mathcal{T}_0 =$ the class of finite trees
- $\mathcal{G}_n =$ the class of graphs which are MSO-interpretable in a tree of $\mathcal{T}_n$
- $\mathcal{T}_{n+1} =$ the class of unfoldings of graphs in G_n

Each structure in the pushdown hierarchy has a decidable MSO-theory.

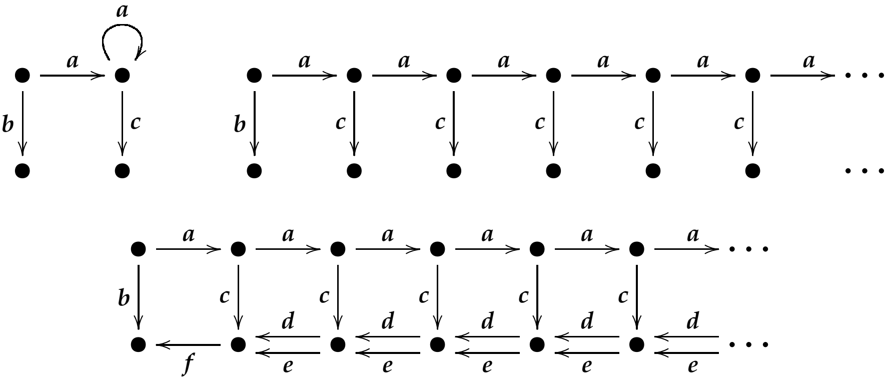
Nontrivial fact:

The sequence $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots$ is strictly increasing.

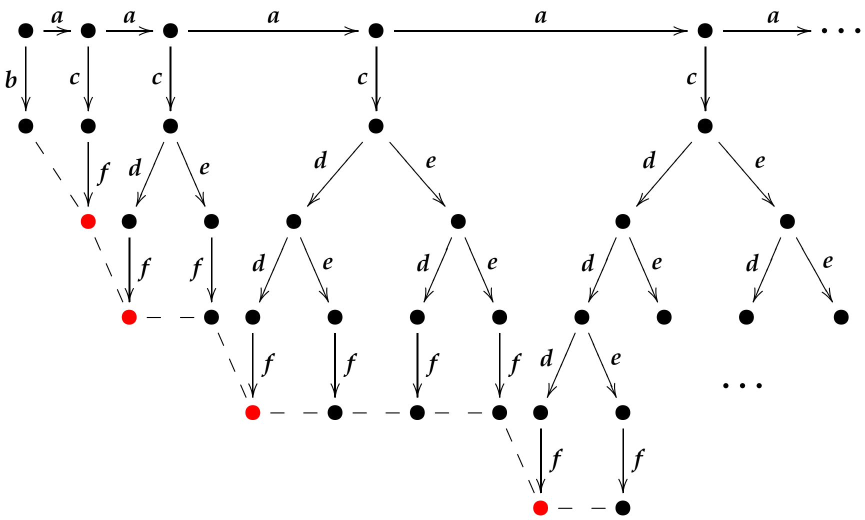
The First Levels

- $\mathcal{G}_0$ is the class of finite graphs.
- $\mathcal{T}_1$ contains the regular trees.
- $\mathcal{G}_1$ contains the prefix-recognizable graphs.

A Finite Graph, a Regular Tree, a PD Graph



Unfolding Again



Interpretation of Bottom Line

The sequence of leaves defines a copy of the successor structure of the natural numbers.

Domain expression: $b + a^*c(d+e)^*f$

Successor relation:

$\overline{b}acf+$

$\overline{f}\overline{e}^*\overline{c}acd^*f+$

$\overline{f}\overline{e}^*\overline{d}ed^*f$

Predicate "power of 2": $b + a^*cd^*f$

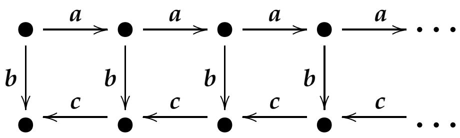
replacing MSO-formulas

Result: $(\mathbb{N}, \mathrm{Succ}, \mathrm{Pow}_2)$ **is a structure in the Caucal hierarchy.**

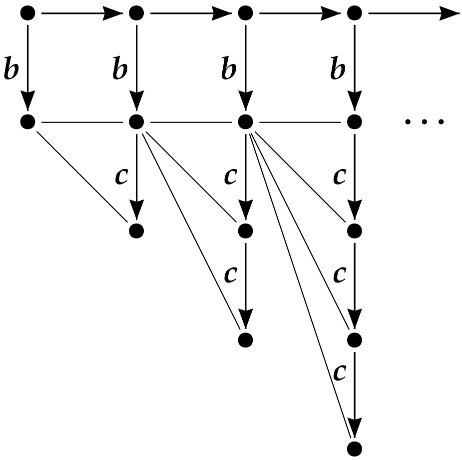
Factorial Predicate

$(\mathbb{N}, \mathbf{Succ}, \mathbf{Fac})$

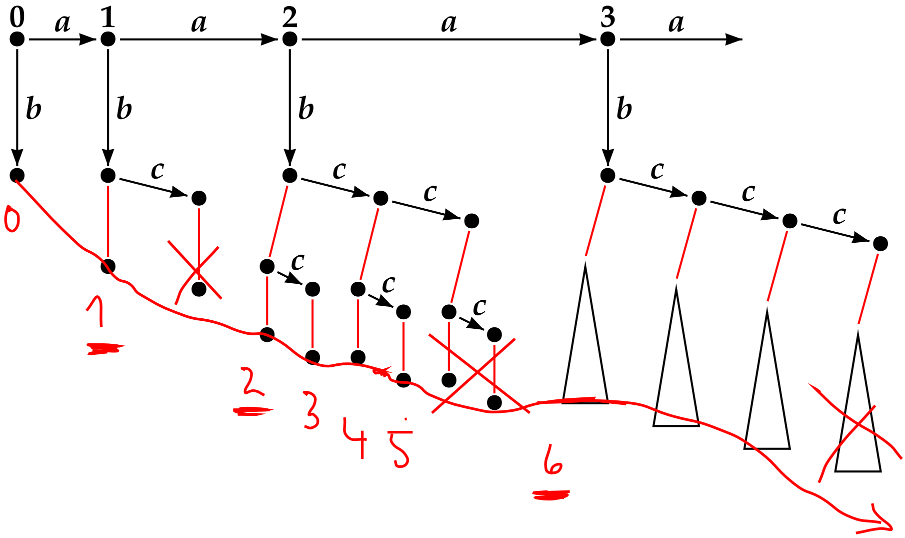
We start as follows:



Continuation: Unfolding and Interpretation



Another Unfolding



Scope of Hierarchy?

The pushdown hierarchy is a very rich class of structures all of which have a decidable MSO-theory.

Open questions:

- **Understand which structures belong to the hierarchy**
- **Compute the smallest level on which a strcuture occurs**