Generalizing “Strategies”

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Infinite Games in Set Theory

Gale-Stewart Game

Play is a single ω-word of bits.

A set $L \subseteq \{0,1\}^\omega$ induces the game Γ_L.

Player 2 wins play γ if $\gamma \in L$.

Variations:

Γ_L^*: Player 2 starts, chooses bit words, Player 1 chooses bits.

Γ_L^{**}: Both players choose bit words (Banach-Mazur game).

The World of the Uncountable

The domain of plays — paths through T_2 — is uncountable: much larger than the countable domain of game positions (vertices of T_2).

The idea of arbitrary sets of ω-words (real numbers) is "new"; it was introduced only 150 years ago.

Also an approach was suggested to compare the size of infinite sets.

Georg Cantor (1845-1918)

Advice from Büchi

The analysis we have presented of the concept of number and counting is one of Cantor's contributions to philosophy (1895–1897). * Go to the library and see how he puts the matter, it makes very interesting reading. The analysis led him to his famous infinite cardinal numbers and ordinal numbers, and to the general idea of isomorphism types of structures. For a long time his friend Dedekind was the only one who listened. In Dedekind (1888) you find recursive definitions for addition and multiplication, do *. So here is the birth of recursion theory, and the subject is very much influenced by Cantor's thoughts,

Determinacy

A game is determined if either Player 1 or Player 2 has a winning strategy.

Central question in set theory: For which L is Γ_L (or Γ_L^* or Γ_L^{**}) determined?

First intuition:

1. For very large L, Player 2 should win.
2. For very small L, Player 1 should win.

“Very small” can mean “at most countable”.
“Very large” can mean “of same cardinility as $\{0,1\}^\omega$.

If L is countable ($L = \{\gamma_0, \gamma_1, \gamma_2, \ldots\}$) then Player 1 wins: His i-th bit makes the play different from γ_i.

Cantor Topology

Cantor's CH says: Each set $L \subseteq \{0,1\}^\omega$ is either very small or very large (i.e., countable or of cardinality $|\{0,1\}^\omega|$).

The set $\{0,1\}^\omega$ is equipped with a topology,

leading to a classification into "simpler" and "more complicated" sets,

using the following metric d:

$$d(\alpha,\beta) = \begin{cases} 0 & \text{, if } \alpha = \beta \\ \frac{1}{2^n} \text{ for smallest } n \text{ with } \alpha(n) \neq \beta(n) & \text{, if } \alpha \neq \beta \end{cases}$$

L is open if it is a union of $\frac{1}{2^n}$-neighbourhoods

Open sets are "simple".

Neighbourhoods and Open Sets

$d(\alpha, \beta) \geq \frac{1}{2^n} \Leftrightarrow$ **for some** $i \leq n : \alpha(i) \neq \beta(i)$

thus

$$d(\alpha, \beta) < \frac{1}{2^n} \Leftrightarrow \underbrace{\alpha(0) \dots \alpha(n)}_{\alpha[0,n]} = \underbrace{\beta(0) \dots \beta(n)}_{\beta[0,n]}$$

Consequence

The $\frac{1}{2^n}$**(**$= \varepsilon$**)-neighbourhood of** $\alpha \in \{0,1\}^\omega$ **is the set**

$$\{\beta \in \mathbb{B}^\omega \mid \beta[0,n] = \alpha[0,n]\}$$

in other words: $\alpha[0,n] \cdot \{0,1\}^\omega$

L **is open if** $L = W \cdot \{0,1\}^\omega$ **for some** $W \subseteq \{0,1\}^*$**.**

"The conceptual distance to finite-word languages is 1".

Closed Sets

An $L \subseteq \{0,1\}^\omega$ is closed iff its complement is open.

For a closed set L a set W of finite words exists with $\alpha \in L$ iff all prefixes of α are in W

The closed sets capture the "abstract safety conditions".

The open sets capture the "abstract guarantee conditions" (reachability).

Cantor-Bendixson Theorem

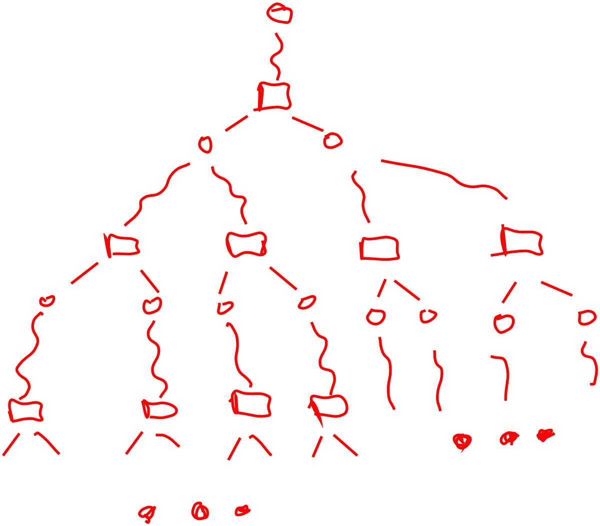
A non-empty closed set $L \subseteq \{0,1\}^\omega$ is either countable or contains a perfect set, i.e., a closed set without isolated points.

Second case gives a copy of the binary tree inside T_2

Consequence:

For closed L, the game Γ_L^* is determined.

A Copy of T_2



Non-Determinacy

(AC) There is a set L such that Γ_L is not determined.

Central idea: Find a winning condition L such that

- **whatever strategy f_2 Player 2 applies, Player 1 can respond by a strategy f_1 such that Player 2 loses:** $\langle f_1, f_2 \rangle \notin L$
- **whatever strategy f_1 Player 1 applies, Player 2 can respond by a strategy f_2 such that Player 1 loses:** $\langle f_1, f_2 \rangle \in L$

We need a systematic way to go through all possible strategies and in this way build up the desired L.

Applying AC, We use transfinite induction over the space of strategies. Ordinals $< \mathfrak{c} = |\mathbb{R}|$ suffice.

Towards a Non-Determined Game

Define for $\xi < \mathfrak{c}$ sets L_ξ and M_ξ with the following properties

- $M_\xi \cap L_\xi = \emptyset$
- $|M_\xi|, \ |L_\xi| < \mathfrak{c}$
- $\forall \eta < \xi \ \exists f \ (\langle f, f_{2,\eta} \rangle \in M_\xi)$ and $\exists g \ (\langle f_{1,\eta}, g \rangle \in L_\xi)$

Let $M_0 = L_0 = \emptyset$

For limit numbers ξ set $M_\xi = \bigcup_{\eta<\xi} M_\eta$ and $L_\xi = \bigcup_{\eta<\xi} L_\eta$

For a successor ordinal ξ consider $f_{1,\xi}$

Choose g such that the play $\langle f_{1,\xi}, g \rangle$ differs from all plays in the previously defined sets L_η, M_η. This is possible since $|\bigcup_{\eta<\xi}(L_\eta \cup M_\eta)| < \mathfrak{c}$ and $|\{\langle f_{1,\xi}, g \rangle \mid g \text{ strategy for } 2\}| = \mathfrak{c}$.

Add the play $\langle f_{1,\xi}, g \rangle$ to the L_η-sets and thus obtain L_ξ

Non-Determinacy

For $f_{2,\xi}$ choose f analogously and obtain M_ξ

Given sets L_ξ and M_ξ as above,

let $L := \bigcup_{\xi<\mathfrak{c}} L_\xi$.

Then the game Γ_L is not determined.

Borel Hierarchy

The Borel hierarchy over $\{0,1\}^\omega$ is built up from the open sets and the closed sets by alternating applications of countable intersections and countable unions.

Define for $n \geq 1$ the classes Σ_n, Π_n of ω-languages:

Σ_1	$:=$	**class of open sets** $L \subseteq \{0,1\}^\omega$
Π_1	$:=$	**class of closed sets** $L \subseteq \{0,1\}^\omega$
Σ_{n+1}	$:=$	**class of countable unions** $L = \bigcup_{i \geq 0} L_i$ **with** $L_i \in \Pi_n$
Π_{n+1}	$:=$	**class of countable intersections** $L = \bigcap_{i \geq 0} L_i$ **with** $L_i \in \Sigma_n$

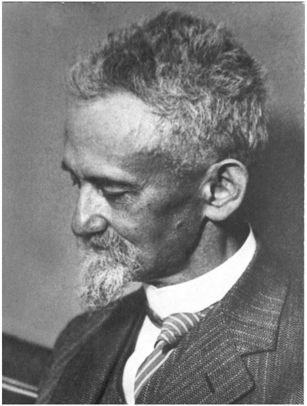
Comments

We have introduced the first levels with indices by natural numbers (the "finite Borel hierarchy").

The classification extends to transfinite (countable) ordinals.

Hausdorff used a different notation for the levels.

G	**for** Σ_1	**("Gebiet")**
F	**for** Π_1	**("ferme")**
G_δ	**for** Π_2	
F_σ	**for** Σ_2 **etc.**	



Felix Hausdorff (1868-1942)

Martin's Theorem

If L is a Borel set then Γ_L is determined.

A regular ω-language over $\{0, 1\}$ can be represented as a Boolean combination of ω-languages defined by formulas $\forall x \exists y \varphi(X, y)$ where y is bounded in y.

Such ω-languages are in the class Π_2

So a regular ω-language is a Boolean combination of Π_2-sets — thus it belongs to the class $\Sigma_3 \cap \Pi_3$

Consequence: Regular games are determined.

Intermediate Summary

1. The dichotomy
   "A set of reals is either very small or very large"
   corresponds to a determinacy result.

2. The dichotomy is true for closed sets (Cantor-Bendixson) but in general a set theoretic hypothesis (CH).

3. The Borel hierarchy starts the open and closed sets, and it gives determined games (Martin's Theorem)

4. The regular games are all determined.

5. But there are exotic non-determined games (assuming AC).

Strategies with Delay

An Example

We look at Gale-Stewart games specified by regular ω-languages.

The players are called I (Input) and O (Output).

A winning condition for Player O:

Player O wins the play $\binom{a_0}{b_0}\binom{a_1}{b_1}\binom{a_2}{b_2}\ldots$ **if**

$b_i = a_{i+1}$ **for all** i

Player I wins by choosing $a_{i+1} \neq b_i$**.**

$$\begin{array}{llllll} \text{Player I:} & 0 & 0 & 0 & 1 & \cdots \\ & \neg\nearrow & \neg\nearrow & \neg\nearrow & & \\ \text{Player O:} & 1 & 1 & 0 & 1 & \cdots \end{array}$$

Games with Delay

We study how the possibility to win is improved for Player O if he is allowed to defer his moves.

- **Theoretical motivation:**
  A function $F : \alpha \mapsto \beta$ where each b_i is determined by $\alpha[0, j]$ for some j is continuous in the Cantor space.
- **Practical motivation: In distributed systems, signal transmission may dissolve the synchronization in the model of Gale-Stewart games.**

Question:

1. **Can we decide whether Player O wins with some delay?**
2. **If the answer is "yes", then how much delay is needed?**

Games with Delay

We represent regular ω-languages by deterministic parity automata.

The delay game $\Gamma_f(\mathcal{A})$ is induced by:

1. **A deterministic parity automaton $\mathcal{A}$**
2. **A function $f : \mathbb{N} \to \mathbb{N}_+$ (called delay function)**

Meaning of f: Player I must choose a word u_i of length $f(i)$

Example: Let $f(i) := i + 1$ for all $i \in \mathbb{N}$

$$\textbf{Player I:}\quad \overbrace{0}^{|u_0|=f(0)}\quad \overbrace{0 \quad 1}^{|u_1|=f(1)}\quad \overbrace{0 \quad 0 \quad 0}^{|u_2|=f(2)}\quad \cdots$$

$$\textbf{Player O:}\quad 1 \quad 0 \quad 1 \quad \cdots$$

The delay here is unbounded.

Degrees of Delay

Player O wins $\Gamma_f(\mathcal{A})$ for some f iff Player O wins by a uniformly continuous function.

Functions of different delay:

1. **Finite delay (possibly unbounded): Any function $f : \mathbb{N} \to \mathbb{N}_+$**
2. **Bounded delay: There exists i_0 such that $f(i) = 1$ for all $i > i_0$.**
3. **Constant delay: $f(0) = d$ and $f(i) = 1$ for all $i > 0$**

Bounded delay can be reduced to constant delay.

(Given a function f of bounded delay, define $g(0) := f(0) + \ldots + f(i_0)$.)

Results

- Given: Deterministic parity automaton $\mathcal{A}$
- Question: Is there a function f such that Player O wins $\Gamma_f(\mathcal{A})$?

F. Hosch, L. Landweber (first ICALP 1972):

- One can decide whether a regular game is solvable with constant delay and determine the minimal necessary delay.

Holtmann, Kaiser, Th. (FoSSaCS 2010, LMCS 2012)

- Let $\mathcal{A}$ be a DPA over $\{0,1\}^2$. The problem whether $L(\mathcal{A})$ is solvable with finite delay is in 2ExpTime$(|\mathcal{A}|)$.
  $L(\mathcal{A})$ is solvable with finite delay iff it is solvable with constant delay d, for some $d \in 2\mathrm{Exp}(|\mathcal{A}|)$.

Sketch of Proof

- Given: Deterministic parity automaton $\mathcal{A}$
- Question: Is there a function f such that Player O wins $\Gamma_f(\mathcal{A})$?

We consider the opposite: Does Player I win $\Gamma_f(\mathcal{A})$ for all f?
Proof strategy:

1. Introduce the "block game"
   - Relax the number of bits Player I can choose in each move.
   - Show that the block game is "equivalent" to the original game.
2. Introduce the "semigroup game"
   - A move of a player is a "behavior" of $\mathcal{A}$, but not a word.
   - Show "equivalence" to block game, and vice versa.

Step 1 – The Block Game

The block game Γ'_f is played as follows:

- **Player I is one move ahead of Player O (compared to Γ_f).**
- **Player I chooses a length for u_i (and v_i) in the interval $[f(i), 2f(i)]$.**

Example:

	$f(0) \leq \lvert u_0 \rvert \leq 2f(0)$	$f(1) \leq \lvert u_1 \rvert \leq 2f(1)$	$f(2) \leq \lvert u_2 \rvert \leq 2f(2)$	$f(3) \leq \lvert u_3 \rvert \leq 2f(3)$	
Player I:	u_0	u_1	u_2	u_3	$\cdots$
Player O:	v_0	v_1	v_2	$\cdots$	
	$\lvert v_0 \rvert = \lvert u_0 \rvert$	$\lvert v_1 \rvert = \lvert u_1 \rvert$	$\lvert v_2 \rvert = \lvert u_2 \rvert$		

Lemma: The following are equivalent:

1. **For all f: Player I wins the game Γ_f.**
2. **For all f: Player I wins the block game Γ'_f.**

Step 2 – The Moves of Player O

Idea: Define the moves of the players to be the possible "behaviors" of $\mathcal{A}$.

Define $\binom{u_1}{v_1} \sim \binom{u_2}{v_2}$ if and only if for all $q \in Q$

1. $\delta^*(q, \binom{u_1}{v_1}) = \delta^*(q, \binom{u_2}{v_2})$
2. On the associated path (cf. item 1) the same maximal color is seen.

Note: Each $\sim$-equivalence class can be identified with a $Q \times Q$-matrix over a finite domain.

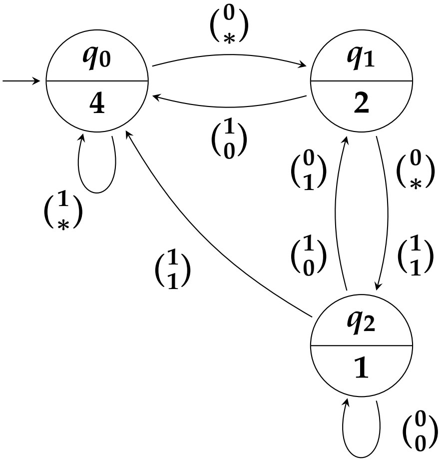
Consequence: The equivalence relation $\sim$ has finite index, i.e., finitely many equivalence classes.

Plan: Take the $\sim$-equivalence classes as moves of Player O.

Example

$$\mu\binom{0\ 1}{0\ 0} = \begin{pmatrix} 4 & \bot & \bot \\ \bot & 2 & \bot \\ \bot & 2 & \bot \end{pmatrix}$$

$$\mu\binom{0\ 0\ 1\ 1}{0\ 1\ 1\ 0} = \begin{pmatrix} 4 & \bot & \bot \\ \bot & 2 & \bot \\ \bot & 2 & \bot \end{pmatrix}$$



The definition of $\sim$ means:

$$\binom{u_1}{v_1} \sim \binom{u_2}{v_2} \Leftrightarrow \mu\binom{u_1}{v_1} = \mu\binom{u_2}{v_2}$$

Let $\mathcal{A}$ be a DPA over $\{0,1\}^2$. All equivalence classes of the relation $\sim$ are regular $*$-languages computable from $\mathcal{A}$.

Step 2 – The Moves of Player I

Problem: If Player I chooses u, then Player O must answer by a class $[\binom{u}{.}]$.

Define $u \approx u'$ iff for each $\sim$-class $\mathcal{C}$

$$\exists v : \binom{u}{v} \in \mathcal{C} \iff \exists v' : \binom{u'}{v'} \in \mathcal{C}$$

The equivalence relation $\approx$ has finite index.

Lemma: Let $\mathcal{A}$ be a DPA over $\{0,1\}^2$. All equivalence classes of the relation $\approx$ are regular $*$-languages effectively computable from $\mathcal{A}$.

Plan: Take the $\approx$-equivalence classes as moves of Player I.

The Semigroup Game

- Player I's moves are the $\approx$-equivalence classes (only infinite ones).
- Player O's moves are the $\sim$-equivalence classes.
- So Player I's choices restrict Player O's possible answers.

Example:

Player I:	$[u_0]$	$[u_1]$	$[u_2]$	$[u_3]$	$\cdots$
Player O	$\left[\binom{u_0}{v_0}\right]$	$\left[\binom{u_1}{v_1}\right]$	$\left[\binom{u_2}{v_2}\right]$ $\cdots$		

Winning condition: Player O wins if and only if $\binom{u_0}{v_0}\binom{u_1}{v_1}\binom{u_2}{v_2}\cdots \in L(\mathcal{A})$.

A Proposition

For all f, Player I wins the block game Γ'_f iff Player I wins the semigroup game.

Simulate a winning strategy for Player I in both directions.

	Block Game		Semigroup Game
Player I:	$f(i) \leq \lvert u_i \rvert \leq 2f(i)$	$\xleftrightarrow{\text{Simulate}}$	$\lvert [u_i] \rvert = \infty$

Task: Estimate the lengths of the words in infinite $\approx$-equivalence classes.

End of Proof

Define $f \sqsupseteq g :\Leftrightarrow f(i) \geq g(i)$ for all $i \in \mathbb{N}$

Lemma: The following are equivalent:

1. **For all f: Player I wins the block game Γ'_f.**
2. **$\exists f_0 \forall f$ ($f \sqsupseteq f_0 \Longrightarrow$ Player wins the block game Γ'_f)**

We need to establish the simulation only for $f \sqsupseteq f_0$.

If $\mathcal{A}$ has n states and m colors, then each $\approx$-equivalence class is recognizable by a DFA with at most $n' := 2^{(mn)^{2n}}$ states.

The function $f_0 := n'$ works.

Summary and Perspective

- **For regular specifications, solvability with finite delay is decidable.**
- **Doubly exponential constant delay is sufficient.**

What about context-free specifications?

- **The problem becomes undecidable for games specified by deterministic parity pushdown automata.**
  In this case, unbounded delay may be necessary, and the corresponding delay function f may have a non-elementary growth.
  (Fridman, Löding, Zimmermann, CSL 2011)